J. Appl. Maths Mechs, Vol. 58, No. 6, pp. 1085–1088, 1994

0021–8928/94 $24.00+0.00

Pergamon

0021–8928(94)00138-3

SYSTEMS WITH AFTER-EFFECT AND RELAXATION†

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(Received 17 June 1993)

A system whose state is described by a scalar parameter is considered. The parameter undergoes relaxation from the initial to a critical value. As soon as the latter is attained, the system is instantaneously brought back into the standard state an the relaxation process begins again. Because relaxation can be described by an equation with delay, each relaxation cycle that follows is different from the previous ones, in general. Some properties of the mathematical model under consideration are established. In particular, conditions are given, under which the long-term behaviour of the system becomes asymptotically periodic.

We will consider a process in a system with after-effect described by a functional-differential delay equation (DDE) with relaxation and impulse support. The latter means that there is a "tension functional" determined by the state of the system (and, in general, its history) and which changes only up to a certain critical value. As soon as this value is attained, the state is altered rapidly and the continuous process then resumes until the tension functional reaches the critical value again, and so on. A mathematical model of such a process is provided by an impulse DDE with impulse times not prescribed in advance, as is usually the case (see, for example, [1, Section 4. 6]), but determined by reaching the corresponding hypersurface in the "Krasovskii space" associated with the given DDE. As far as we are aware, no impulse DDEs of this type other than ordinary impulse differential equations have been considered.

Below, for an autonomous scalar impulse DDE with a simple tension functional, we present a precise formulation of the problem and obtain general properties of the solutions as well as sufficient conditions for their asymptotic periodicity. It will also be interesting to consider a more general situation.

Suppose the initial DDE is

$$\dot{x}(t)=f(x(t+\theta_1),\ldots,x(t+\theta_m),x_t)\quad (x_t(\theta):=x(t+\theta),-h\leqslant\theta\leqslant 0) \tag{1}$$

where $0 < h < \infty$, all $\theta_j \in [-h, 0]$, f: $\mathbf{R}^m \times K[-h, 0] \to \mathbf{R}$, and $K[-h, 0]$ is the set of left-continuous functions $[-h, 0] \to \mathbf{R}$ having a finite number of discontinuities, all of which are of the first kind. We assume that f satisfies the following Lipschitz condition

$$\begin{aligned}
&|f(u_1,\ldots,u_m,\varphi)-f(v_1,\ldots,v_m,\psi)|\leqslant M_1|u_1-v_1|+\ldots+\\
&+M_m|u_m-v_m|+M_0\int_{-h}^{0}|\varphi(\theta)-\psi(\theta)|d\theta\\
&(\forall u_1,\ldots,u_m,v_1,\ldots,v_m\in\mathbf{R},\ \ \varphi,\psi\in K[-h,0];\ M_1,\ldots,M_m,M_0=\text{const}\geqslant 0)
\end{aligned}$$

Note that the right-hand side of (1) covers situations which are natural from the viewpoint of applications. However, the form $\dot{x}(t) = g(x_t)$ of the equation gives rise to a difficulty concerned with the domain of the functional g.

Let the tension functional J, its critical value J_*, an the jump ΔJ, which occurs when this value is reached, be equal to, respectively

$$J(\varphi)=\varphi(0)\ \ (\forall\varphi\in K[-h,0]),\quad J_*(\varphi)=0,\quad \Delta x=a=\text{const}>0 \tag{2}$$

Equation (1) together with conditions (2) define the impulse DDE.

†Prikl. Mat. Mekh. Vol. 58, No. 6, pp. 141–144, 1994.

Under the initial condition

$$x_{t_0} = \varphi \in K[-h,0] \tag{3}$$

the solution $\bar{x}$ of (1)–(3) can be constructed as follows. If in an arbitrary interval $[t_0, t_1)$ the solution x of problem (1), (3) (x is assumed to be continuous in $[t_0, t_1]$ and $\dot{x}$ can have a finite number of discontinuities of the first kind) has no zeros, then $\bar{x}(t) \equiv x(t)$ $(t_0 \leqslant t \leqslant t_1)$. But if $x(t) \neq 0$ $(t_0 \leqslant t \leqslant t_1)$, $x(t_1) = 0$ (in particular, if $\varphi(0) = 0$, then $t_1 = t_0$), then, for $t > t_1$, $\bar{x}$ is constructed as a solution of (1) with initial point t_1 and initial function defined for $t \leqslant t_1$ by the solution $\bar{x}$ already constructed, but with initial value $x(t_1^+) = a$. For $t > t_1$ this solution x can be extended up to the first instant $t_2 > t_1$ when $x(t_2) = 0$ (if such an instant occurs), after which we set $\bar{x}(t_2^+)$ again, and so on.

The existence and uniqueness of the solution of (1)–(3) in a sufficiently small time interval with left end t_0 can be proved in the standard way. Setting, for brevity

$$f_0 := |f(0,\dots,0,0)|, \quad M := M_1 + \dots + M_m + hM_0$$

from the Lipschitz condition we obtain the following estimate in the domain of existence of $\bar{x}$

$$|f(\bar{x}(t+\theta_1),\dots,\bar{x}(t+\theta_m),\ \bar{x}_t)| \leqslant f_0 + M \sup|\bar{x}_t|$$

From the latter we find, also in a standard way, that

$$|x(t)| \leqslant \begin{cases} \max\{\sup|\varphi|,\ a\} e^{M(t-t_0)} + f_0 M^{-1}[e^{M(t-t_0)} - 1] & (M > 0) \\ \max\{\sup|\varphi|,\ a\} + f_0(t-t_0) & (M = 0) \end{cases}$$

It follows that in each finite interval (t_0, t) of the existence of $\bar{x}$ the value of $|\dot{\bar{x}}|$ has an upper limit between the discontinuities of $\bar{x}$, and so there may be only a finite number of such points in $[t_0, t)$. This means that $\bar{x}$ can be extended as far as required, i.e. the following result has been proved.

Theorem 1. Under the above assumptions, a unique solution of problem (1)–(3) exists in any interval $[t_0, \bar{t})$ $(t_0 < \bar{t} \leqslant \infty)$.

Remark. Under the hypotheses of Theorem 1, the solution of (1)–(3) may, in general, fail to depend continuously on the initial function of φ.

The picture becomes much simpler if we assume additionally that

$$-\delta := \sup\{f(u_1,\dots,u_m,\psi):\ u_1,\dots,u_m \in [0,d],\ 0 \leqslant \inf\psi,\ \sup\psi \leqslant d\} < 0 \tag{4}$$

for some $d \in [a, \infty)$. The following assertion is true.

Theorem 2. Let the inequality (4) and the hypotheses of Theorem 1 be satisfied and let $0 \leqslant \inf\varphi$, $\sup\varphi \leqslant d$. Then, for the solution x of (1)–(3), we have

$$0 \leqslant x(t) < \max\{a, \varphi(t_0)\} \quad (t_0 < t < \infty)$$

$$x(t) < a \quad (t_0 + \max\{\delta^{-1}[\varphi(t_0) - a], 0\} < t < \infty)$$

This solution has an infinite sequence $t_1(\geqslant t_0) < t_2 < \dots$ of discontinuities, for which

$$a(f_0 + dM)^{-1} \leqslant t_{k+1} - t_k \leqslant a\delta^{-1} \quad (k = 1, 2, \dots)$$

$$t_{k+1} - t_k \geqslant h_* \ \ (k\!:\ t_k \geqslant t_0 + h + \max\{\delta^{-1}[\varphi(t_0) - a], 0\}; \quad h_* := a(f_0 + aM)^{-1})$$

If $\sup\varphi \leqslant a$, then the latter inequality holds for all $k \geqslant 1$.

To study the asymptotic behaviour of the solution x of (1)–(3) as $t \to \infty$ we introduce an analogue of the Poincaré return map, namely, an operator P which assigns x_{t_2} to any function $\varphi \in K[-h, 0]$ with $\varphi(0) = 0 = \min\varphi$, $\sup\varphi \leqslant d$. The following lemma gives an estimate of the Lipschitz constant of this operator.

Lemma 1. Let the hypothesis of Theorem 2 with $d = a$ be satisfied for the initial functions φ^1 and φ^2, let $\varphi^1(0) = \varphi^2(0) = 0$, and let

$$aM \leqslant \delta, \quad h \leqslant h_*, \tag{5}$$

Then

$$\sup|P\varphi^2 - P\varphi^1| \leqslant aM\delta^{-2}(f_0 + aM + \delta)p \quad (p := \sup|\varphi^2 - \varphi^1|) \tag{6}$$

Proof. We can set $t_0^1 = t_0^2 = 0$ without loss of generality. Then, as long as there are no discontinuities, we have

$$|\dot{x}^2(t) - \dot{x}^1(t)| \leqslant M \max\left\{p, \max_{0<\tau\leqslant t} \Delta_{21}(\tau)\right\} \quad (\Delta_{21}(\tau) := |x^2(\tau) - x^1(\tau)|)$$

for $t > 0$. Since $x^1(0^+) = x^2(0^+) = a$, it follows that

$$\Delta_{21}(t) \leqslant Mpt \quad (0 \leqslant t \leqslant \min\{M^{-1}, t_2^1, t_2^2\}) \tag{7}$$

But, by Theorem 2 and the first inequality in (5)

$$t_2^i \leqslant a\delta^{-1} \leqslant M^{-1} \quad (i = 1, 2)$$

To fix our ideas, support $t_2^1 \leqslant t_2^2$. Then we find from (7) that

$$\Delta_{21}(t) \leqslant Mpt \quad (0 \leqslant t \leqslant t_2^1)$$

However, from Theorem 2 and the first inequality in (5) it follows that $t_2^1 \geqslant h$. Therefore we have

$$|x^2(\theta + t_2^2) - x^1(\theta + t_2^1)| \leqslant |x^2(\theta + t_2^2) - x^2(\theta + t_2^1)| + \Delta_{21}(\theta + t_2^1) \leqslant (t_2^2 - t_2^1)(f_0 + aM) + Mpa\delta^{-1}$$

for $-h < \theta \leqslant 0$. But if $t_2^2 - t_2^1 > 0$, then $x^2(t_2^1) > 0$, and so

$$t_2^2 - t_2^1 = \frac{t_2^2 - t_2^1}{x^2(t_2^1) - x^2(t_2^2)}[x^2(t_2^1) - x^2(t_2^2)] \leqslant \delta^{-1}Mpt_2^1 \leqslant Mpa\delta^{-2}$$

Substituting this estimate into the previous one, we obtain (6). Lemma 1 is proved.

If the return map is a contraction, the system described by an impulse DDE of the type under consideration has a unique limit cycle, which is stable. This is stated in the following theorem.

Theorem 3. Let condition (4) and the hypotheses of Theorem 1 be satisfied, and let $h \leqslant h_*$ and

$$2aM \leqslant (f_0^2 + 2f_0\delta + 5\delta^2)^{1/2} - f_0 - \delta \tag{8}$$

Then the impulse DDE (1), (2) has a unique periodic solution $t \mapsto \bar{x}(t)$ in the region $0 \leqslant x(t) < a$ apart from an arbitrary shift in t. Under the hypotheses of Theorem 2, the solution tends to $\bar{x}$ asymptotically as $t \to \infty$ in the following sense: $t_k = kT + \text{const} + O(e^{-\alpha t})$, where T is the period of $\bar{x}$, $\alpha > 0$, and

$$\sup|x(\cdot + t_k) - \bar{x}(\cdot + \bar{t})|_{[0, t_{k+1} - t_k] \cap [0, T]} = O(e^{-\alpha k})$$

as $k \to \infty$, where $\bar{x}(\bar{t}) = 0$.

Proof. Let x be a solution of problem (1)–(3) under the hypotheses of Theorem 2. Consider the sequence of functions

$$\theta \mapsto x_k(\theta) := x(\theta + t_k) \quad (k = 1, 2, \ldots;\ t_{k-1} - t_k < \theta \leqslant 0)$$

By Theorem 2 and the inequality $h \leqslant h_*$, for all sufficiently large k, each of these functions can be regarded as the initial function for the next one. Setting for $\varphi^i(\theta) = x_{k+i-1}(\theta)$ $(i = 1, 2)$, we can apply Lemma 1, which implies that

$$\sup_{-h<\theta\leqslant 0} |x_{k+2}(\theta) - x_{k+1}(\theta)| \leqslant aM\left(\frac{1}{\delta} + \frac{1}{f_0} + \frac{aM}{\delta f_0}\right) \sup_{-h<\theta\leqslant 0} |x_{k+1}(\theta) - x_k(\theta)|$$

But inequality (8) implies that the coefficient on the right-hand side is less than one. This means that the sequence $\{x_k\}$ of uniformly continuous functions on $(-h, 0]$ converges uniformly to $\bar{x}$, which is also uniformly continuous on $(-h, 0]$, the rate of convergence being exponential.

Letting $k \to \infty$ in the formula $x_{t_{k+1}} = Px_{t_k}$, we find that if $\bar{x}$ is taken to be the initial function, then the corresponding solution of (1)–(3) turns out to be periodic. It follows that $\bar{x}$ can be extended onto the whole t-axis to obtain a periodic solution of the impulse DDE (1), (2). The period of $\bar{x}$ is equal to $\lim_{k\to\infty}(t_{k+1} - t_k)$. Now, since the impulse DDE (1), (2) is autonomous, it follows that $x(t)$ "approaches" $\bar{x}(t)$ asymptotically as $t \to \infty$. The periodic solution is unique because Lemma 1 could be applied to two such solutions, which completes the proof of Theorem 3.

The inequalities $h \leqslant h_*$ and (8), which ensure the existence of a stable limit cycle, are quite restrictive, and it would be desirable to relax them and, in particular, to investigate the case $h = \infty$. It would also be interesting to obtain the impulse DDE under consideration as the limit of a more realistic singularly perturbed system with high but finite rate of variation of the state as the critical situations are reached.

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Translated by T.J.Z.